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# Skew polynomial rings of formal triangular matrix rings

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Let $R$, $S$ be rings with unity and $M$ be a unital $(R,S)$-bimodule. In this paper we give a description of homomorphisms and skew derivations of the formal triangular matrix ring $T=\left(\begin{smallmatrix} R & M \\ 0 & S\end{smallmatrix}\right)$, and apply it to provide a triangular representation of the skew polynomial ring $T[z;\theta,d]$. Also we introduce some special mappings on modules which are generalization of ring homomorphisms and skew derivations. We characterize the ring endomorphisms of $T$, when $T[z;\theta,d]$ has a triangular representation. These results are applied to introduce the notion of skew polynomial modules and present some results and examples concerning this notion.



## 1. Introduction

Throughout this paper all rings are associative with unity, all modules are unital, and all ring homomorphisms preserve the unity element. Suppose that $R$ is a ring and $\alpha$ is a ring endomorphism of $R$. Recall that an *$\alpha$-derivation*, or a *skew derivation*, of $R$ is an additive map $\delta: R\to R$ such that $\delta(rr')=\alpha(r)\delta(r')+\delta(r)r'$, for all $r,r'\in R$. In case $\alpha$ is the identity map, $\delta$ is called a *derivation* of $R$. For each fixed element $a\in R$, the mapping $I_a: R\to R$ given by $I_a(r)=\alpha(r)a-ar$ for all $r\in R$, is an $\alpha$-derivation of $R$. $I_a$ is called an *inner $\alpha$-derivation*. Recall that for an invertible element $c\in R$, the ring automorphism $\varphi_c: R\to R$ given by $\varphi_c(r)=crc^{-1}$ for each $r\in R$, is called an *inner automorphism*.

Let $R$ be a ring, $\alpha$ a ring endomorphism of $R$, and $\delta$ an $\alpha$-derivation on $R$. Recall that $R[x;\alpha,\delta]$ is the *skew polynomial ring over* $R$, or an *Ore extension of* $R$, whose elements are the polynomials over $R$, the addition is defined as usual and the multiplication subject to the relation $xr=\alpha(r)x+\delta(r)$ for any $r\in R$. In case $\alpha$ is the identity map, we have the ring of differential operators $R[x;\delta]$, if $\delta$ is the zero map, we have the skew polynomial ring $R[x;\alpha]$ and if $\alpha$ is the identity map and $\delta$ is the zero map, we have the ordinary polynomial ring $R[x]$.

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Recall that a *formal triangular matrix ring* $\mathcal{T}(R, M, S)$ is a ring of the form

$$\mathcal{T}(R,M,S) := \left\{ \begin{pmatrix} r & m \\ 0 & s \end{pmatrix} : r \in R,\ m \in M,\ s \in S \right\},$$

under the usual matrix operations, where $R$ and $S$ are rings and $M$ is an $(R, S)$-bimodule. We denote the elements of $\mathcal{T}(R, M, S)$ by $(r, m, s)$. The most important examples of formal triangular matrix rings are upper triangular matrices over a ring $R$, block upper triangular matrix algebras, nest algebras over a real or a complex Banach space $X$ or a Hilbert space $H$, respectively and $n$-triangular algebras (see [14]). Also, formal triangular matrix rings have appeared often in the study of the representation theory of rings and algebras (for instance, see [2,6–8,12,13,17] and [18]). Note that every invertible element of $\mathcal{T}(R, M, S)$ can be written as a 3-tuple $(r, m, s)$ such that $r$ and $s$ are invertible elements of $R$ and $S$, respectively and $m$ is any element of $M$. In this case $(r, m, s)^{-1} = (r^{-1}, -r^{-1}ms^{-1}, s^{-1})$.

The representations of a ring as a formal triangular matrix ring provide an important tool in the study of the structure of a wide range of rings, so it is natural and very interesting to find some conditions under which a ring has a formal triangular matrix representation. Birkenmeier et al. in [4] developed the theory of generalized triangular matrix representations in an abstract setting which is a generalization of formal triangular matrix representation. A large class of ring extensions which have generalized triangular matrix representations is investigated by Birkenmeier et al. in [5] (see also [19]). By [9], a differential polynomial extension of a formal triangular matrix ring has a triangular representation in terms of the formal triangular matrix ring.

In this paper we study the structure of a skew polynomial ring over a formal triangular matrix ring $T := \mathcal{T}(R, M, S)$, for suitable ring endomorphisms, and show that this ring extension has a formal triangular matrix representation. Indeed, if $d$ is a $\theta$-derivation of $T$, where $\theta$ is a special sort of ring endomorphism of $T$, we prove the ring isomorphism:

$$T[z;\theta,d] \cong \mathcal{T}\big(R[x;\alpha,\delta_R], M[y;\gamma,\tau], S[y;\beta,\delta_S]\big),$$

where $R[x; \alpha, \delta_R]$ and $S[y; \beta, \delta_S]$ are the skew polynomial rings over $R$ and $S$, respectively, and $M[y; \gamma, \tau]$ is an $(R[x; \alpha, \delta_R], S[y; \beta, \delta_S])$-bimodule. Accordingly, we introduce the notion of *skew polynomial module* as a generalization of skew polynomial ring and some polynomial modules (see [1,3, 15] and [16]). Also we study the relations between the ring endomorphisms of $T$ and the triangular representations of $T[z; \theta, d]$ as a formal triangular matrix ring.

This paper is organized as follows. In Section 2 we give a description of homomorphisms and skew derivations of formal triangular matrix rings which generalize the results of [9] and apply it to determine the triangular representations of $T[z; \theta, d]$. Also we introduce some special mappings on modules which are generalization of ring homomorphisms and skew derivations. In Section 3, we investigate the structure of $T[z; \theta, d]$. In particular, we consider the triangular representation of $T[z; \theta, d]$ in terms of the formal triangular matrix ring, where $\theta$ is a special ring endomorphism of $T$. Furthermore, we characterize the ring endomorphisms of $T$, when $T[z; \theta, d]$ has a triangular representation. Finally, in Section 4 our previous results are applied to introduce the notion of skew polynomial modules and present some results and examples concerning this notion.

The following terminology is used throughout this article.

Let $R$ be a ring and $M$ be a left $R$-module, define the left annihilator of $M$ by $l.ann_A M = \{r \in R\colon\ rM = \{0\}\}$. We write $E_{ij}$ for the matrix units and $aE_{ij}$ for the matrix with the $a$ at the $(i, j)$-entry and 0 in all other entries. For any matrix $G$, we denote $(G)_{ij}$ for the $(i, j)$-entry of $G$. $id_U$ stands for the identity map on the set $U$. We use $I_a$ and $\varphi_c$ for the inner $\alpha$-derivations and inner automorphisms, respectively. Also, we employ lower case letters to denote elements in rings and modules in the abstract setting and upper case letters to denote elements in formal triangular matrix rings.

## 2. Homomorphisms and skew derivations on formal triangular matrix rings

In this section we study the homomorphisms and skew derivations on formal triangular matrix rings.

**Proposition 2.1.** *Let $T_1 = \mathcal{T}(R, M, S)$, $T_2 = \mathcal{T}(A, N, B)$ and $\theta : T_1 \to T_2$ be a mapping. Then the following are equivalent*:

(i) $\theta = \varphi_D \circ \Lambda$, *where $\varphi_D$ is an inner automorphism of $T_2$ induced by some invertible matrix $D \in T_2$ and the mapping $\Lambda : T_1 \to T_2$ is given by*

$$\Lambda(r, m, s) = \big(\alpha(r), \gamma(m), \beta(s)\big),$$

*for some ring homomorphisms* (*isomorphisms*) $\alpha : R \to A$, $\beta : S \to B$ *and an additive* (*bijective additive*) *mapping $\gamma : M \to N$ satisfying $\gamma(rs) = \alpha(r)\gamma(m)$ and $\gamma(ms) = \gamma(m)\beta(s)$ for every $r \in R$, $s \in S$, and $m \in M$.*

(ii) *$\theta$ is a ring homomorphism* (*isomorphism*) *such that $\theta(RE_{11}) \subseteq AE_{11} + NE_{12}$ and $\theta(SE_{22}) \subseteq NE_{12} + BE_{22}$.*

(iii) *$\theta$ is a ring homomorphism* (*isomorphism*) *such that $(\theta(E_{11}))_{11} = 1$ and $(\theta(E_{22}))_{22} = 1$.*

**Proof.** (i) $\Rightarrow$ (ii) Clear.

(ii) $\Rightarrow$ (iii) Since $I = \theta(I) = \theta(E_{11}) + \theta(E_{22})$, we find

$$\big(\theta(E_{11})\big)_{11} + \big(\theta(E_{22})\big)_{11} = 1 \quad \text{and} \quad \big(\theta(E_{11})\big)_{22} + \big(\theta(E_{22})\big)_{22} = 1.$$

By (ii) we obtain $(\theta(E_{11}))_{22} = 0$ and $(\theta(E_{22}))_{11} = 0$. These relations give the desired conclusion.

(iii) $\Rightarrow$ (i) From $I = \theta(E_{11}) + \theta(E_{22})$ and (iii) we have

$$\big(\theta(E_{11})\big)_{22} = 0, \qquad \big(\theta(E_{22})\big)_{11} = 0, \qquad \big(\theta(E_{11})\big)_{12} = -\big(\theta(E_{22})\big)_{12}.$$

Let $n_0 = (\theta(E_{11}))_{12} \in N$ and $D = (1, -n_0, 1) \in T_2$. Then $D$ is invertible and the mapping $\Lambda = \varphi_{D^{-1}} \circ \theta$ from $T_1$ into $T_2$ is a ring homomorphism such that $\Lambda(E_{11}) = E_{11}$ and $\Lambda(E_{22}) = E_{22}$. Therefore, for each $r \in R$ we have $\Lambda(rE_{11}) = \Lambda(rE_{11})E_{11} = \alpha(r)E_{11}$ for some mapping $\alpha : R \to A$. Applying $\Lambda$ to $(r + r')E_{11} = rE_{11} + r'E_{11}$ and $rr'E_{11} = rE_{11}r'E_{11}$, for each $r, r' \in R$, we see that $\alpha$ is a ring homomorphism.

Similarly, by applying $\Lambda$ to $sE_{22} = E_{22}sE_{22}$, $(s + s')E_{22} = sE_{22} + s'E_{22}$ and $ss'E_{22} = sE_{22}s'E_{22}$ (for each $s, s' \in S$), we get a ring homomorphism $\beta : S \to B$ such that $\Lambda(sE_{22}) = \beta(s)E_{22}$.

Now let $m, m' \in M$. Applying $\Lambda$ to $mE_{12} = E_{11}mE_{12} = mE_{12}E_{22}$ and $(m + m')E_{12} = mE_{12} + m'E_{12}$, we obtain an additive mapping $\gamma : M \to N$ such that $\Lambda(mE_{12}) = \gamma(m)E_{12}$. Therefore, for each $r \in R$ and $m \in M$ we have

$$\gamma(rm)E_{12} = \Lambda(rmE_{12}) = \Lambda(rE_{11})\Lambda(mE_{12}) = \alpha(r)E_{11}\gamma(m)E_{12} = \alpha(r)\gamma(m)E_{12}.$$

So $\gamma(rm) = \alpha(r)\gamma(m)$, for each $r \in R$ and $m \in M$. By a similar computation we find that $\gamma(ms) = \gamma(m)\beta(s)$, for each $s \in S$ and $m \in M$. Therefore we have $\theta = \varphi_D \circ \Lambda$ and $\Lambda(r, m, s) = (\alpha(r), \gamma(m), \beta(s))$, for each $r \in R, m \in M, s \in S$, where $\alpha, \gamma, \beta$ and $D$ satisfy the required conditions. If $\theta$ is a ring isomorphism, then $\Lambda = \varphi_{D^{-1}} \circ \theta$ is a ring isomorphism. So, for each $a \in A$, there exist $(r, m, s) \in T_1$ such that $\Lambda(r, m, s) = aE_{11}$, and hence $\alpha(r) = a$. Therefore $\alpha$ is surjective. Now assume that $\alpha(r) = 0$, so $\Lambda(rE_{11}) = 0$. Since $\Lambda$ is injective, we have $r = 0$. Thus $\alpha$ is injective and hence it is a ring isomorphism. Similarly, we can show that $\beta$ is a ring isomorphism an $\gamma$ is a bijective mapping. $\square$

The proof of this proposition leads immediately to next corollary.

**Corollary 2.2.** *Let $T_1$, $T_2$ and $\theta$ be as above. Then the following are equivalent*:

(i) $\theta(r, m, s) = (\alpha(r), \gamma(m), \beta(s))$, *where $\alpha : R \to A$, $\beta : S \to B$ are ring homomorphisms* (*isomorphisms*) *and $\gamma : M \to N$ is a additive* (*bijective additive*) *mapping such that*

$$\gamma(rm)=\alpha(r)\gamma(m) \quad \textit{and} \quad \gamma(ms)=\gamma(m)\beta(s),$$

*for each* $r\in R$, $s\in S$, $m\in M$.

(ii) $\theta$ *is a ring homomorphism (isomorphism) such that* $\theta(RE_{11})\subseteq AE_{11}$ *and* $\theta(SE_{22})\subseteq BE_{22}$.

(iii) $\theta$ *is a ring homomorphism (isomorphism) such that* $\theta(E_{11})=E_{11}$ *and* $\theta(E_{22})=E_{22}$.

**Remark 2.3.** Not all ring homomorphisms (isomorphisms) on formal triangular matrix rings are in the form of Proposition 2.1. For example, let $T=\mathcal{T}(R,M,S)$ and let $\alpha:R\to S$, $\beta:S\to R$ be nonzero ring homomorphisms. Then the mapping $\theta:T\to T$ given by $\theta(r,m,s)=(\beta(s),0,\alpha(r))$ is easily seen to be a ring homomorphism that does not satisfy the conditions of Proposition 2.1.

The above proposition extends a result of Ghosseiri [10, Theorem 2.1], which states that if $R$ and $S$ are reduced rings (i.e., rings without nonzero nilpotent elements) whose idempotents are central, and $M$ is an $(R,S)$-bimodule such that $l.ann_R(M)=\{0\}$, then all automorphisms on $T=\mathcal{T}(R,M,S)$ are in the form of Proposition 2.1(i).

In view of Proposition 2.1(i), we have the following definition.

**Definition 2.4.** Let $R$, $S$, $A$, $B$ be rings, $M$ be an $(R,S)$-bimodule, $N$ be an $(A,B)$-bimodule, and $\alpha:R\to A$, $\beta:S\to B$ be ring homomorphisms. Then an additive mapping $\gamma:M\to N$ is called a *bimodule homomorphism relative to* $(\alpha,\beta)$, if $\gamma(rm)=\alpha(r)\gamma(m)$ and $\gamma(ms)=\gamma(m)\beta(s)$ for each $r\in R, s\in S$ and $m\in M$.

Note that the conditions on $\gamma$ in above definition are equivalent to saying that it is a homomorphism of $(R,S)$-bimodules from $M$ to $N$, where the latter is taken with the $(R,S)$-bimodule structure induced by its given $(A,B)$-bimodule structure and the homomorphisms $\alpha:R\to A$ and $\beta:S\to B$.

Relative bimodule homomorphisms are used in studying skew polynomial rings over formal triangular matrix rings and in defining skew polynomial modules. The following example provides various types of relative bimodule homomorphisms.

**Example 2.5.** Let $M$ be an $(R,S)$-bimodule and $N$ be an $(A,B)$-bimodule.

(i) If $A=R$ and $B=S$, and $\gamma:M\to N$ is a bimodule homomorphism, then $\gamma$ is a bimodule homomorphism relative to $(id_R, id_S)$.

(ii) If $R$ is considered as an $(R,R)$-bimodule, $\gamma:R\to R$ is a bimodule homomorphism and $\alpha,\beta$ are ring endomorphisms on $R$, then $\alpha\circ\gamma\circ\beta:R\to R$ is a bimodule homomorphism relative to $(\alpha\circ\beta,\alpha\circ\beta)$. Taking $\gamma=\beta=id_R$ one observes that every ring endomorphism $\alpha$ of $R$ is a bimodule homomorphism relative to $(\alpha,\alpha)$.

(iii) Let $T=\mathcal{T}(R,M,S)$. We make $R$ into a unitary $(T,T)$-bimodule by defining $(r,m,s)r':=rr'$, $r'(r,m,s):=r'r$, for each $r,r'\in R$, $s\in S$ and $m\in M$. Suppose that $\theta$ is a ring endomorphism of $T$, given by $\theta(r,m,s)=(\alpha(r),\gamma(m),\beta(s))$ as in Proposition 2.1(i). Then $\alpha:R\to R$ is a bimodule homomorphism relative to $(\theta,\theta)$.

We continue by characterizing the skew derivations of formal triangular matrix rings.

**Proposition 2.6.** *Suppose that* $\theta$ *is a ring endomorphism on the triangular matrix ring* $T=\mathcal{T}(R,M,S)$ *given by* $\theta(r,m,s)=(\alpha(r),\gamma(m),\beta(s))$, *where* $\gamma$ *is a bimodule homomorphism relative to* $(\alpha,\beta)$ *on* $M$. *Then* $d:T\to T$ *is a* $\theta$*-derivation if and only if* $d=\bar{d}+I_G$, *where* $I_G$ *is the inner* $\theta$*-derivation with* $G=d(E_{11})\in ME_{12}$ *and* $\bar{d}$ *is given by* $\bar{d}(r,m,s)=(\delta_R(r),\tau(m),\delta_S(s))$, *where* $\delta_R:R\to R$ *is an* $\alpha$*-derivation,* $\delta_S:S\to S$ *is a* $\beta$*-derivation and* $\tau:M\to M$ *is an additive mapping such that*

$$\tau(rm)=\alpha(r)\tau(m)+\delta_R(r)m,$$

$$\tau(ms)=\gamma(m)\delta_S(s)+\tau(m)s,$$

*for each* $r\in R$, $s\in S$ *and* $m\in M$.

**Proof.** Suppose that $d : T \to T$ is a $\theta$-derivation. Applying $d$ to the equations: $E_{11} = E_{11}^2$, $E_{11} + E_{22} = I$, $mE_{12} = mE_{12}E_{22} = E_{11}mE_{12}$, $rE_{11} = E_{11}rE_{11} = rE_{11}E_{11}$ and $sE_{22} = E_{22}sE_{22} = sE_{22}E_{22}$ ($r \in R$, $s \in S$, $m \in M$), and using Corollary 2.2 we get the mappings $\delta_R : R \to R$, $\delta_S : S \to S$, $\tau : M \to M$ and some $m_0 \in M$ such that $d(E_{11}) = m_0E_{12}$, $d(E_{22}) = -m_0E_{12}$, $d(rE_{11}) = \delta_R(r)E_{11} + \alpha(r)m_0E_{12}$, $d(sE_{22}) = \delta_S(s)E_{22} - m_0sE_{12}$ and $d(m_0E_{12}) = \tau(m)E_{12}$ for all $r \in R$, $s \in S$ and $m \in M$.

Now by applying $d$ to $(r + r')E_{11} = rE_{11} + r'E_{11}$, $(s + s')E_{22} = sE_{22} + s'E_{22}$, $(m + m')E_{12} = mE_{12} + m'E_{12}$, $rr'E_{11} = rE_{11}r'E_{11}$, $ss'E_{22} = sE_{22}s'E_{22}$, $rmE_{12} = rE_{11}mE_{12}$ and $msE_{12} = mE_{12}sE_{22}$ for each $r, r' \in R$, $s, s' \in S$ and $m, m' \in M$, we deduce that $\delta_R$ is an $\alpha$-derivation, $\delta_S$ is a $\beta$-derivation and $\tau$ is an additive mapping such that satisfies $\tau(rm) = \alpha(r)\tau(m) + \delta_R(r)m$ and $\tau(ms) = \gamma(m)\delta_S(s) + \tau(m)s$. Finally, for each $(r, m, s) \in T$ we have

$$d(r, m, s) = \bigl(\delta_R(r), \tau(m), \delta_S(s)\bigr) + \bigl(\alpha(r)m_0 - m_0s\bigr)E_{12} = \bar{d}(r, m, s) + I_{m_0E_{12}}(r, m, s),$$

where $G = m_0E_{12} = d(E_{11})$ and $\bar{d}(r, m, s) = (\delta_R(r), \tau(m), \delta_S(s))$.

Conversely, if $d$ satisfies the given conditions, it is easy to show that $d$ is a $\theta$-derivation of $T$. □

The proposition above motivates the next definition.

**Definition 2.7.** Let $M$ be an $(R, S)$-bimodule, $\gamma : M \to M$ be a bimodule homomorphism relative to $(\alpha, \beta)$, $\delta_R : R \to R$ be an $\alpha$-derivation, and $\delta_S : S \to S$ be a $\beta$-derivation. The additive mapping $\tau : M \to M$ is called a *generalized $\gamma$-derivation relative to* $(\delta_R, \delta_S)$, or a *skew generalized derivation relative to* $(\delta_R, \delta_S)$, if $\tau(rm) = \alpha(r)\tau(m) + \delta_R(r)m$ and $\tau(ms) = \gamma(m)\delta_S(s) + \tau(m)s$, for each $r \in R$, $s \in S$ and $m \in M$.

In case $\alpha = id_R$, $\beta = id_S$ and $\gamma = id_M$, then $\delta_R$ and $\delta_S$ are derivations and $\tau$ is just called a *generalized derivation relative to* $(\delta_R, \delta_S)$. Thus a generalized derivation relative to $(\delta_R, \delta_S)$ on $M$ is any additive mapping $\tau : M \to M$ that satisfies the identities: $\tau(rm) = r\tau(m) + \delta_R(r)m$ and $\tau(ms) = m\delta_S(s) + \tau(m)s$, for each $r \in R$, $s \in S$ and $m \in M$, where $\delta_R$ and $\delta_S$ are derivations of $R$ and $S$, respectively. Moreover putting $\delta_R = 0 = \delta_S$, one notes that a generalized derivation on a bimodule $M$ is also a generalization of a bimodule homomorphism on $M$. The notion of generalized derivation was introduced in [9].

By Proposition 2.6, we see that if $\tau : M \to M$ is a generalized $\gamma$-derivation relative to $(\delta_R, \delta_S)$, then $d : T \to T$ given by $d(r, m, s) = (\delta_R(r), \tau(m), \delta_S(s))$, is a $\theta$-derivation, where $T = \mathcal{T}(R, M, S)$ and $\theta(r, m, s) = (\alpha(r), \gamma(m), \beta(s))$.

The notion of skew generalized derivation on modules extends some classes of mappings on rings or modules. The following example provides various types of skew generalized derivations.

**Example 2.8.** Let $M$ be an $(R, S)$-bimodule and $\gamma : M \to M$ be a bimodule homomorphism relative to $(\alpha, \beta)$.

(i) Let $a \in R$ and $b \in S$. Then the map $\tau_{a,b} : M \to M$ defined by $\tau_{a,b}(m) = am - \gamma(m)b$ is a generalized $\gamma$-derivation relative to $(I_{-a}, I_{-b})$.
(ii) The mapping $\tau(m) = \gamma(m) - m$ is a generalized $\gamma$-derivation relative to $(\delta_R, \delta_S)$, where $\delta_R = \alpha(r) - r$ is an $\alpha$-derivation on $R$ and $\delta_S = \beta(s) - s$ is a $\beta$-derivation on $S$.
(iii) If $\beta = id_S$, then $\gamma$ is a generalized $\gamma$-derivation relative to $(I_0, I_0)$, where $I_0$ is the zero derivation. In particular if $\alpha = id_R$ and $\beta = id_S$, then every module endomorphism $\gamma : M \to M$ is a generalized $\gamma$-derivation relative to $(I_0, I_0)$.
(iv) If $R$ is considered as an $(R, R)$-bimodule, then every $\alpha$-derivation $\delta : R \to R$ is a generalized $\alpha$-derivation relative to $(\delta, \delta)$, where $\alpha$ is considered as a bimodule homomorphism on $R$ relative to $(\alpha, \alpha)$.

## 3. Skew polynomial rings of formal triangular matrix rings

In this section we study the skew polynomial extension construction on formal triangular matrix rings. If $T = \mathcal{T}(R, M, S)$, then we provide a triangular representation of the skew polynomial ring $T[z; \theta, d]$.

First, we need the following lemma.

**Lemma 3.1.** *(See [11, p. 37].) Let $R$ be a ring with a ring endomorphism $\alpha$ and an $\alpha$-derivation $\delta$. Set $S = R[x; \alpha, \delta]$.*

(i) *Suppose that there exists a ring $T$, a ring homomorphism $\phi : R \to T$, and an element $y \in T$ such that $y\phi(r) = \phi(\alpha(r))y + \phi(\delta(r))$ for all $r \in R$. Then there exists a unique ring homomorphism $\psi : S \to T$ such that $\psi|_R = \phi$ and $\psi(x) = y$. In particular, $\psi(\sum_i r_i x^i) = \sum_i \phi(r_i) y^i$.*
(ii) *If $S' = R[x'; \alpha, \delta]$, there is a unique ring isomorphism $\psi : S \to S'$ such that $\psi(x) = x'$ and $\psi|_R$ is the identity map on $R$.*

**Proposition 3.2.** *Let $R$ be a ring, $\alpha$ a ring endomorphism of $R$, and $\delta$ an $\alpha$-derivation on $R$.*

(i) *If $\delta_1 : R \to R$ is an $\alpha$-derivation such that $\delta = \delta_1 + I_a$ for some inner $\alpha$-derivation $I_a$ of $R$, then*

$$R[x; \alpha, \delta] \cong R[z; \alpha, \delta_1],$$

*by an isomorphism sending $x$ to $z - a$.*
(ii) *If $c \in R$ is invertible, then $c\delta$ is a $\varphi_c \circ \alpha$-derivation and*

$$R[x; \alpha, \delta] \cong R[z; \varphi_c \circ \alpha, c\delta],$$

*by an isomorphism sending $x$ to $c^{-1}z$.*

**Proof.** (i) Put $y = z - a \in R[z; \alpha, \delta_1]$ and consider the mapping $\phi : R \to R[z; \alpha, \delta_1]$ given by $\phi(r) = r$. We have, $y\phi(r) = (z - a)r = \alpha(r)z + \delta_1(r) - ar = \alpha(r)(z - a) + \delta_1(r) + I_a(r) = \phi(\alpha(r))y + \phi(\delta(r))$. So by Lemma 3.1, there exists a unique ring homomorphism $\psi : R[x; \alpha, \delta] \to R[z; \alpha, \delta_1]$ such that

$$\psi(r) = r \quad \text{and} \quad \psi(x) = z - a.$$

Similarly, applying Lemma 3.1 to $\phi : R \to R[x; \alpha, \delta]$, with $\phi(r) = r$ and $y = x + a \in R[x; \alpha, \delta]$, we arrive at a unique ring homomorphism $\psi' : R[z; \alpha, \delta_1] \to R[x; \alpha, \delta]$ such that

$$\psi'(r) = r \quad \text{and} \quad \psi'(z) = x + a.$$

So we have $\psi \circ \psi'(r) = r$ for all $r \in \mathcal{R}$ and $\psi \circ \psi'(z) = z$. Therefore, by Lemma 3.1 we have $\psi \circ \psi' = id_{R[z;\alpha,\delta_1]}$. Similarly we get $\psi' \circ \psi = id_{R[x;\alpha,\delta]}$. Thus $\psi$ is a ring isomorphism and we have $R[x; \alpha, \delta] \cong R[z; \alpha, \delta_1]$.

(ii) A routine computation shows that $c\delta$ is a $\varphi_c \circ \alpha$-derivation.

Define $\phi : R \to R[x; \alpha, \delta]$ by $\phi(r) = r$ and let $y = cx$. So

$$y\phi(r) = c\alpha(r)x + c\delta(r) = \phi\big(\varphi_c \circ \alpha(r)\big)y + \phi\big(c\delta(r)\big).$$

By Lemma 3.1, there exists a unique ring homomorphism $\psi : R[z; \varphi_c \circ \alpha, c\delta] \to R[x; \alpha, \delta]$ such that

$$\psi(r) = r \quad \text{and} \quad \psi(z) = cx.$$

Similarly, we obtain a unique ring homomorphism $\psi' : R[x; \alpha, \delta] \to R[z; \varphi_c \circ \alpha, c\delta]$ such that

$$\psi'(r) = r \quad \text{and} \quad \psi'(x) = c^{-1}z.$$

Hence, from Lemma 3.1 we get $\psi \circ \psi' = id_{R[x;\alpha,\delta]}$ and $\psi' \circ \psi = id_{R[z;\varphi_c\circ\alpha,c\delta]}$. Therefore $\psi$ is a ring isomorphism and we have $R[x; \alpha, \delta] \cong R[z; \varphi_c \circ \alpha, c\delta]$. $\square$

**Remark 3.3.** Let $R$ be a ring, $\alpha$ a ring endomorphism of $R$, and $\delta$ an $\alpha$-derivation on $R$. By Proposition 3.2, we have $R[x; \alpha, \delta] \cong R[\theta; \varphi_c \circ \alpha, c\delta - I_a]$, where $c \in R$ is invertible, $\varphi_c : R \to R$ is an inner automorphism and $I_a (a \in R)$ is an inner $\varphi_c \circ \alpha$-derivation on $R$.

If $T$ is a ring and $e \in T$ is an idempotent such that $e'Te = \{0\}$, where $e' = 1 - e$, then $R := Te$ and $S := e'T$ are rings with the addition and multiplication of $T$, but different unities, and $M := eTe'$ is an additive subgroup of $T$ which is also an $(R, S)$-bimodule. We have $eTe = Te$, $e'Te' = e'T$ and $e, e'$ are the identity elements of $R$ and $S$ respectively.

**Proposition 3.4.** *(See [4, Proposition 1.3].) Let $T$ be a ring and $e \in T$ be an idempotent such that $e'Te = \{0\}$ where $e' = 1 - e$. Assume that $R := Te$, $S := e'T$ and $M := eTe'$. Then the mapping $g : T \to \mathcal{T}(R, M, S)$ given by $g(t) = (te, ete', e')$, is a ring isomorphism.*

The following proposition determines some equivalent conditions relating the structure of a ring endomorphism of $T = \mathcal{T}(R, M, S)$, $\theta : T \to T$ and a skew polynomial ring over $T$.

**Proposition 3.5.** *Let $T = \mathcal{T}(R, M, S)$, $\theta : T \to T$ be a ring endomorphism, and $d : T \to T$ be a $\theta$-derivation. Then the following conditions are equivalent*:

(i) *$\theta(r, m, s) = (\alpha(r), \gamma(m), \beta(s))$, where $\gamma : M \to M$ is a bimodule homomorphism relative to $(\alpha, \beta)$ and $d(r, m, s) = (\delta_R(r), \tau(m), \delta_S(s))$, where $\tau : M \to M$ is a generalized $\gamma$-derivation relative to $(\delta_R, \delta_S)$.*
(ii) *$E_{11}zE_{22} = 0$ and $E_{22}zE_{11} = 0$ in $T[z; \theta, d]$.*
(iii) *$E_{11}z = zE_{11}$ in $T[z; \theta, d]$.*
(iv) *$E_{22}z = zE_{22}$ in $T[z; \theta, d]$.*

*If any of above conditions holds, then in $T[z; \theta, d]$, for each positive integer $n$, we have*

$$(E_{11}z)^n = E_{11}z^n = z^n E_{11} \quad \textit{and} \quad (E_{22}z)^n = E_{22}z^n = z^n E_{22},$$

*and we have the ring isomorphism*

$$T[z; \theta, d] \cong \mathcal{T}\big(T[z; \theta, d]E_{11}, E_{11}T[z; \theta, d]E_{22}, E_{22}T[z; \theta, d]\big).$$

**Proof.** (i) $\Rightarrow$ (ii) Using the definitions of $\theta$ and $d$, we have

$$E_{11}zE_{22} = E_{11}\theta(E_{22})z + d(E_{22}) = E_{11}E_{22}z + d(E_{22}) = 0.$$

Similarly, $E_{22}zE_{11} = 0$.

(ii) $\Rightarrow$ (iii) We have $zE_{11} = E_{11}zE_{11} + E_{22}zE_{11} = E_{11}zE_{11}$ and $E_{11}z = E_{11}zE_{11} + E_{11}zE_{22} = E_{11}zE_{11}$. So, $E_{11}z = zE_{11}$.

(iii)$\Leftrightarrow$(iv) Assume that $E_{11}z = zE_{11}$. Then $E_{22}z = zI - E_{11}z = zI - zE_{11} = zE_{22}$. Similarly, if $E_{22}z = zE_{22}$, then $E_{11}z = zE_{11}$.

(iii) and (iv) $\Rightarrow$ (i) We have

$$E_{22}z = zE_{22} = \theta(E_{22})z + d(E_{22}).$$

So $\theta(E_{22}) = E_{22}$, $d(E_{22}) = 0$ and $E_{11}z = zE_{11}$. Therefore,

$$E_{11}z = zE_{11} = \theta(E_{11})z + d(E_{11}).$$

Hence $\theta(E_{11}) = E_{11}$ and $d(E_{11}) = 0$. From these relations, Corollary 2.2 and Proposition 2.6 it follows that $\theta$ and $d$ have the desired forms in (i).

Now suppose that any one of the above conditions holds. We proceed by induction on $n$. For $n = 1$ there is nothing to prove. Assume inductively that the result holds for $n$. We have $E_{11}z^{n+1} = z^n E_{11}z = z^{n+1}E_{11}$ and $(E_{11}z)^{n+1} = E_{11}z^n E_{11}z = E_{11}z^n z = E_{11}z^{n+1}$.

So $(E_{11}z)^n = E_{11}z^n = z^n E_{11}$, for each positive integer $n$. Similarly, $(E_{22}z)^n = E_{22}z^n = z^n E_{22}$ $(n \geqslant 1)$.

Now for each $p = \sum_i t_i z^i \in T[z;\theta,d]$, we have

$$E_{22}pE_{11} = \sum_i E_{22}t_i z^i E_{11} = \sum_i E_{22}t_i E_{11}z^i = 0.$$

Therefore, $E_{22}T[z;\theta,d]E_{11} = 0$ and by Proposition 3.4 we get the desired ring isomorphism. □

Now we provide a representation of the skew polynomial ring $T[z;\theta,d]$ in terms of the formal triangular matrix ring.

**Proposition 3.6.** *Let $T = \mathcal{T}(R,M,S)$ and let $\theta(r,m,s) = (\alpha(r),\gamma(m),\beta(s))$ be a ring endomorphism of $T$ satisfying the conditions in Corollary* 2.2(i)*. Suppose that $d : T \to T$ is a $\theta$-derivation given by $d(r,m,s) = (\delta_R(r),\tau(m),\delta_S(s))$, where $\delta_R$ is an $\alpha$-derivation on $R$, $\delta_S$ is a $\beta$-derivation on $S$ and $\tau$ is a generalized $\gamma$-derivation relative to $(\delta_R,\delta_S)$ on $M$. Then we have the ring isomorphism*:

$$T[z;\theta,d] \cong \mathcal{T}\big(R[x;\alpha,\delta_R], M[y;\gamma,\tau], S[y;\beta,\delta_S]\big),$$

*where $R[x;\alpha,\delta_R]$ and $S[y;\beta,\delta_S]$ are skew polynomial rings over $R$ and $S$ respectively, and $M[y;\gamma,\tau]$ is an $(R[x;\alpha,\delta_R], S[y;\beta,\delta_S])$-bimodule such that*:

(i) *$M[y;\gamma,\tau]$ contains $M$ as an $(R,S)$-subbimodule.*
(ii) *For each $m \in M$, we have $xm = \gamma(m)y + \tau(m)$.*
(iii) *Each element $p \in M[y;\gamma,\tau]$ is uniquely written as $p = m_0 + m_1 y + \cdots + m_k y^k$, with $m_i \in M$ and $y^i \in S[y;\beta,\delta_S]$ $(1 \leqslant i \leqslant k)$.*

**Proof.** By Proposition 3.5 we have

$$T[z;\theta,d] \cong \mathcal{T}\big(T[z;\theta,d]E_{11}, E_{11}T[z;\theta,d]E_{22}, E_{22}T[z;\theta,d]\big).$$

So $T[z;\theta,d]$ is isomorphic to a generalized triangular matrix ring. Next we show that

$$T[z;\theta,d]E_{11} \cong R[x;\alpha,\delta_R] \quad \text{and} \quad E_{22}T[z;\theta,d] \cong S[y;\beta,\delta_S].$$

The mapping $\phi : R \to T[z;\theta,d]E_{11}$ defined by $\phi(r) = rE_{11}$ is a ring homomorphism. If $a = zE_{11}$, then by Proposition 3.5 we have

$$\begin{aligned}a\phi(r) &= zE_{11}rE_{11} = \theta(rE_{11})z + d(rE_{11}) = \alpha(r)E_{11}z + \delta_R(r)E_{11} \\ &= \alpha(r)E_{11}zE_{11} + \delta_R(r)E_{11} = \phi\big(\alpha(r)\big)a + \phi\big(\delta_R(r)\big).\end{aligned}$$

So, by Lemma 3.1, there exists a unique ring homomorphism $\psi : R[x;\alpha,\delta_R] \to T[z;\theta,d]E_{11}$ such that

$$\psi(r) = rE_{11}, \qquad \psi(x) = zE_{11} \quad \text{and} \quad \psi\left(\sum_i r_i x^i\right) = \sum_i r_i E_{11} z^i E_{11}.$$

Clearly, $\psi$ is injective. Let $pE_{11} = (\sum_i (r_i, m_i, s_i)z^i)E_{11} \in T[z;\theta,d]E_{11}$. For the element $q = \sum_i r_i x^i \in R[x;\alpha,\delta_R]$, we have $\psi(q) = \psi(\sum_i r_i x^i) = \sum_i r_i E_{11} z^i E_{11} = pE_{11}$. Therefore $\psi$ is surjective. Hence, $\psi$ is a ring isomorphism. Similarly, we get a ring isomorphism $\psi' : S[y;\beta,\delta_S] \to E_{22}T[z;\theta,d]$ such that

$$\psi'(s) = sE_{22}, \qquad \psi'(y) = E_{22}z \quad \text{and} \quad \psi'\left(\sum_i s_i y^i\right) = \sum_i s_i E_{22} z^i.$$

Obviously, $E_{11}T[z;\theta,d]E_{22}$ is a $(T[z;\theta,d]E_{11}, E_{22}T[z;\theta,d])$-bimodule. Therefore, if we identify $M[y;\gamma,\tau]$ with $E_{11}T[z;\theta,d]E_{22}$, in view of the above isomorphisms, $M[y;\gamma,\tau]$ is an $(R[x;\alpha,\delta_R], S[y;\beta,\delta_S])$-bimodule, by the following operations:

$$\left(\sum_i r_i x^i\right)(E_{11}pE_{22}) := \psi\left(\sum_i r_i x^i\right)(E_{11}pE_{22}),$$

and

$$(E_{11}pE_{22})\left(\sum_i s_i y^i\right) := (E_{11}pE_{22})\psi'\left(\sum_i s_i y^i\right).$$

Therefore, $M[y;\gamma,\tau]$ is an $(R,S)$-bimodule, and for each $m \in M$ we have $mE_{12} = E_{11}mE_{12}E_{22} \in M[y;\gamma,\tau]$. Since $ME_{12}$ is an $(R,S)$-bimodule, if upon identifying $M$ with $ME_{12}$, $M$ is an $(R,S)$-subbimodule of $M[y;\gamma,\tau]$. Now we have

$$\begin{aligned}xm &= zE_{11}mE_{12} = \theta(mE_{12})z + d(mE_{12}) = \gamma(m)E_{12}z + \tau(m)E_{12} \\ &= \gamma(m)E_{12}E_{22}z + \tau(m)E_{12} = \gamma(m)y + \tau(m).\end{aligned}$$

By the definition, each element $p \in M[y;\gamma,\tau]$ is of the form

$$p = E_{11}\left(\sum_i (r_i, m_i, s_i)z^i\right)E_{22}.$$

So

$$p = \sum_i E_{11}(r_i, m_i, s_i)E_{22}z^i = \sum_i m_i E_{12}E_{22}z^i = \sum_i m_i y^i.$$

Therefore each element $p \in M[y;\gamma,\tau]$ can be written as $\sum_i m_i y^i$ with $m_i \in M$. If $\sum_i m_i y^i = 0$, then $\sum_i m_i E_{12}(E_{22}z)^i = \sum_i m_i E_{12} z^i = 0$. Thus we have $m_i = 0$, for each $i$. This shows that each element $p \in M[y;\gamma,\tau]$ has a unique representation of the form $p = \sum_i m_i y^i$.

Now, if we consider the identity mapping $id$ on $M[y;\gamma,\tau]$, then we have

$$id(qp)=qp=\psi(q)id(p) \quad \text{and} \quad id\big(pq'\big)=pq'=id(p)\psi'\big(q'\big),$$

for each $p\in M[y;\gamma,\tau]$, $q\in R[x;\alpha,\delta_R]$ and $q'\in S[y;\beta,\delta_S]$. Therefore, since $M[y;\gamma,\tau]=E_{11}T[z;\theta,d]E_{22}$, the identity mapping on $M[y;\gamma,\tau]$ is a bimodule homomorphism relative to $(\psi,\psi')$. Now from Corollary 2.2 and noting that $\psi$, $\psi'$ are ring isomorphisms and $id$ is bijective, we conclude that the mapping

$$\Lambda:\mathcal{T}\big(R[x;\alpha,\delta_R],M[y;\gamma,\tau],S[y;\beta,\delta_S]\big)\to\mathcal{T}\big(T[z;\theta,d]E_{11},E_{11}T[z;\theta,d]E_{22},E_{22}T[z;\theta,d]\big),$$

given by

$$\Lambda\big(q,p,q'\big)=\big(\psi(q),id(p),\psi'\big(q'\big)\big),$$

is a ring isomorphism. This completes the proof. □

Note that the isomorphism given in the above proposition maps $z$ to $(x,0,y)$ and the isomorphism restricted to $\mathcal{T}(R,M,S)$ is the identity map.

The following is our main result.

**Theorem 3.7.** *Let $T=\mathcal{T}(R,M,S)$ and, let $\theta(r,m,s)=(\alpha(r),\gamma(m),\beta(s))$ be a ring endomorphism of $T$ satisfying the conditions in Corollary 2.2(i). Suppose $D\in T$ is invertible and $d$ is a $\varphi_D\circ\theta$-derivation of $T$. Then there exists a ring isomorphism*

$$\psi:T[z;\varphi_D\circ\theta,d]\to\mathcal{T}\big(R[x;\alpha,\delta_R],M[y;\gamma,\tau],S[y;\beta,\delta_S]\big),$$

*where $\delta_R$ is an $\alpha$-derivation on $R$, $\delta_S$ is a $\beta$-derivation on $S$, $\tau$ is a generalized $\gamma$-derivation relative to $(\delta_R,\delta_S)$ on $M$ and $\mathcal{T}(R[x;\alpha,\delta_R],M[y;\gamma,\tau],S[y;\beta,\delta_S])$ is as in Proposition 3.6. Moreover, there exist $m_1,m_2\in M$ such that*

$$\big(\psi\big((1,m_1,1)z-m_2E_{12}\big)\big)_{12}=0.$$

**Proof.** By Proposition 3.2(ii), we have a ring isomorphism $\Lambda_1:T[z;\varphi_D\circ\theta,d]\to T[z';\theta,D^{-1}d]$, where $\Lambda_1(z)=Dz'$. $D^{-1}d$ is a $\theta$-derivation on $T$, so by Proposition 2.6 we have $D^{-1}d=d'+I_G$, where $I_G$ is an inner $\theta$-derivation for some $G\in ME_{12}$ and $d':T\to T$ is given by $d'(r,m,s)=(\delta_R(r),\tau(m),\delta_S(s))$ for an $\alpha$-derivation $\delta_R:R\to R$, a $\beta$-derivation $\delta_S:S\to S$ and a generalized $\gamma$-derivation $\tau:M\to M$ relative to $(\delta_R,\delta_S)$. So from Proposition 3.2(i) we get a ring isomorphism $\Lambda_2:T[z';\theta,D^{-1}d]\cong T[z'';\theta,d']$, where $\Lambda_2(z')=z''-G$. Now by Proposition 3.6 we have a ring isomorphism

$$\Lambda_3:T\big[z'';\theta,d'\big]\to\mathcal{T}\big(R[x;\alpha,\delta_R],M[y;\gamma,\tau],S[y;\beta,\delta_S]\big),$$

where $\Lambda_3(z'')=(x,0,y)$. Also, $\Lambda_1$, $\Lambda_2$ and $\Lambda_3$ leaves every elements of $T$ invariant. So

$$\psi:T[z;\varphi_D\circ\theta,d]\to\mathcal{T}\big(R[x;\alpha,\delta_R],M[y;\gamma,\tau],S[y;\beta,\delta_S]\big),$$

given by $\psi=\Lambda_3\circ\Lambda_2\circ\Lambda_1$ is a ring isomorphism such that $\psi(D^{-1}z+G)=(x,0,y)$. Let $m_1=-(D)_{12}(D)_{22}^{-1}$ and $m_2=-(D)_{11}(G)_{12}$. Thus

$$\psi\big((1,m_1,1)z-m_2E_{12}\big)=\big((D)_{11}x,0,(D)_{22}y\big).$$

This completes the proof of the theorem. □

**Remark 3.8.** In view of the preceding theorem, if $\theta$ is a ring endomorphism of $T = \mathcal{T}(R, M, S)$ which satisfies any of the equivalent conditions in Proposition 2.1, then $T[z;\theta,d]$ has a formal triangular matrix representation as described in Theorem 3.7.

By Ghosseiri [10, Theorem 2.1], when $R$ and $S$ are reduced rings whose idempotents are central and $M$ is an $(R, S)$-bimodule such that $l.ann_R(M) = 0$, then all ring automorphisms of $T = \mathcal{T}(R, M, S)$ are in the form given in Proposition 2.1(i). So in this case, for each ring automorphism $\theta$ of $T = \mathcal{T}(R, M, S)$, the ring $T[z;\theta,d]$ has a formal triangular matrix representation as given in Theorem 3.7.

The converse of Theorem 3.7 is also true:

**Theorem 3.9.** *Let $T = \mathcal{T}(R, M, S)$, $\theta : T \to T$ be a ring endomorphism, and $d : T \to T$ be a $\theta$-derivation. Suppose $\mathcal{T}(A, N, B)$ is a formal triangular matrix ring and $\psi : T[z;\theta,d] \to \mathcal{T}(A, N, B)$ is a ring isomorphism such that $\psi(E_{11}) = E_{11}$, $\psi(E_{22}) = E_{22}$ and for some $m_1, m_2 \in M$*

$$\big(\psi\big((1, m_1, 1)z - m_2E_{12}\big)\big)_{12} = 0. \tag{$*$}$$

*Then $\theta = \varphi_D \circ \Lambda$, where $\varphi_D$ is an inner automorphism induced by some $D \in T$ and $\Lambda : T \to T$ is given by*

$$\Lambda(r, m, s) = \big(\alpha(r), \gamma(m), \beta(s)\big),$$

*for some ring endomorphisms $\alpha : R \to R$, $\beta : S \to S$ and a bimodule homomorphism $\gamma : M \to M$ relative to $(\alpha, \beta)$.*

*Also we have the ring isomorphisms $A \cong R[x;\alpha,\delta_R]$ and $B \cong S[y;\beta,\delta_S]$, where $\delta_R$ is an $\alpha$-derivation of $R$ and $\delta_S$ is a $\beta$-derivation of $S$. Moreover, if we consider $N$ as an $(R[x;\alpha,\delta_R], S[y;\beta,\delta_S])$-bimodule via this ring isomorphisms, then $N$ is isomorphic to some module $M[y;\gamma,\tau]$, as an $(R[x;\alpha,\delta_R], S[y;\beta,\delta_S])$-bimodule, where $\tau : M \to M$ is a generalized $\gamma$-derivation relative to $(\delta_R, \delta_S)$, and $M[y;\gamma,\tau]$ is an $(R[x;\alpha,\delta_R], S[y;\beta,\delta_S])$-bimodule as described in Proposition* 3.6.

**Proof.** Let $D = (1, -m_1, 1) \in T$, so $D$ is invertible and $D^{-1} = (1, m_1, 1)$. If we set $p = D^{-1}z - m_2E_{12}$, then by hypothesis we get $\psi(p) = aE_{11} + bE_{22}$, for some $a \in A$ and $b \in B$. We have

$$\psi(E_{11}p) = E_{11}(aE_{11} + bE_{22}) = aE_{11} = (aE_{11} + bE_{22})E_{11} = \psi(pE_{11}).$$

Since $\psi$ is injective, it follows that $E_{11}p = pE_{11}$. On the other hand,

$$\begin{aligned} pE_{11} &= \big(D^{-1} - m_2E_{12}\big)E_{11} = D^{-1}\theta(E_{11})z + D^{-1}d(E_{11}), \\ E_{11}p &= E_{11}D^{-1}z - E_{11}m_2E_{12} = (E_{11} + m_1E_{12})z + mE_{12}. \end{aligned}$$

So by the above relations we have $D^{-1}\theta(E_{11}) = E_{11} + m_1E_{12}$. Similarly, $\psi(pE_{22}) = \psi(E_{22}p)$ and we get $D^{-1}\theta(E_{22}) = E_{22}$. Now we have

$$D^{-1}\theta(E_{11})D = (E_{11} + m_1E_{12})D = E_{11} \quad \text{and} \quad D^{-1}\theta(E_{22})D = E_{22}D = E_{22}.$$

Thus, $\varphi_{D^{-1}} \circ \theta(E_{11}) = E_{11}$ and $\varphi_{D^{-1}} \circ \theta(E_{22}) = E_{22}$. Therefore, $\varphi_{D^{-1}} \circ \theta$ is a ring endomorphism of $T$ satisfying the conditions of Corollary 2.2(iii). So $\varphi_{D^{-1}} \circ \theta(r, m, s) = (\alpha(r), \gamma(m), \beta(s))$, where $\alpha$ and $\beta$ are ring endomorphisms on $R$ and $S$, respectively and $\gamma$ is a bimodule homomorphism relative to $(\alpha, \beta)$ on $M$. Define $\Lambda : T \to T$ by $\Lambda(r, m, s) = (\alpha(r), \gamma(m), \beta(s))$. Then $\theta = \varphi_D \circ \Lambda$, and the result follows.

Now by Theorem 3.7 we have the following ring isomorphism

$$T[z;\theta,d] \cong \mathcal{T}\big(R[x;\alpha,\delta_R], M[y;\gamma,\tau], S[y;\beta,\delta_S]\big).$$

This isomorphism maps the elements

$$E_{11}, E_{22} \in \mathcal{T}\big(R[x;\alpha,\delta_R], M[y;\gamma,\tau], S[y;\beta,\delta_S]\big)$$

to the elements $E_{11}, E_{22} \in T[z;\theta,d]$, respectively. So by hypothesis we have the following ring isomorphism

$$\Phi : \mathcal{T}\big(R[x;\alpha,\delta_R], M[y;\gamma,\tau], S[y;\beta,\delta_S]\big) \to \mathcal{T}(A,N,B),$$

where $\Phi(E_{11}) = E_{11}$ and $\Phi(E_{22}) = E_{22}$. Therefore, by Corollary 2.2, it follows that

$$\Phi(q_1,p,q_2) = \big(\alpha'(q_1), \gamma'(p), \beta'(q_2)\big),$$

where $\alpha' : R[x;\alpha,\delta_R] \to A$ and $\beta' : S[y;\beta,\delta_S] \to B$ are ring isomorphisms and $\gamma' : M[y;\gamma,\tau] \to N$ is a bijective additive mapping such that

$$\gamma'(q_1 p) = \alpha'(q_1)\gamma'(p) \quad \text{and} \quad \gamma'(pq_2) = \gamma'(p)\beta'(q_2),$$

for $q_1 \in R[x;\alpha,\delta_R]$, $q_2 \in S[y;\beta,\delta_S]$ and $p \in M[y;\gamma,\tau]$. Now the following module multiplications make $N$ into an $(R[x;\alpha,\delta_R], S[y;\beta,\delta_S])$-bimodule:

$$q_1 n := \alpha'(q_1)n \quad \text{and} \quad nq_2 := n\beta'(q_2).$$

Consequently, $\gamma'$ is an $(R[x;\alpha,\delta_R], S[y;\beta,\delta_S])$-bimodule isomorphism and $N$ is isomorphic to $M[y;\gamma,\tau]$, as an $(R[x;\alpha,\delta_R], S[y;\beta,\delta_S])$-bimodule. □

By the proof of above theorem, if we replace the condition $(*)$ with $(\psi(z))_{12} = 0$ in Theorem 3.9, then $\theta$ is in the form of Corollary 2.2(i). So in this case we have the converse of Proposition 3.6.

In the following example we show that the condition $(*)$ in Theorem 3.9, is not superfluous.

**Example 3.10.** Assume that $M$ is an $(R,S)$-bimodule. Then we make $M$ into an $(R\times S, R\times S)$-bimodule by defining:

$$(r,s)m := rm \quad \text{and} \quad m(r,s) := ms \quad \big((r,s) \in R\times S, m \in M\big).$$

Let $T = \mathcal{T}(R\times S, M, R\times S)$ and define $\theta : T \to T$, by

$$\theta\big((r,s), m, (r',s')\big) = \big((r,s'), 0, (r',s')\big).$$

Then $\theta$ is a ring endomorphism of $T$. For each positive integer $n$ we have $\theta^n(E_{11}) = (1,0)E_{11}$. So in the ring $T[z;\theta]$ we have

$$\begin{aligned} E_{22}\big((r,s), m, (r',s')\big)z^n E_{11} &= E_{22}\big((r,s), m, (r',s')\big)\theta^n(E_{11})z^n \\ &= E_{22}\big((r,s), m, (r',s')\big)\big((1,0)E_{11}\big)z^n = 0. \end{aligned}$$

Thus, by Proposition 3.4 we have the following ring isomorphism

$$T[z;\theta] \cong \mathcal{T}\big(T[z;\theta]E_{11}, E_{11}T[z;\theta]E_{22}, E_{22}T[z;\theta]\big),$$

where the elements $E_{11}, E_{22} \in T[z;\theta]$ are mapped to the elements $E_{11}, E_{22} \in \mathcal{T}(T[z;\theta]E_{11}, E_{11}T[z;\theta]E_{22}, E_{22}T[z;\theta])$, respectively. But

$$\big(\theta(E_{11})\big)_{11} = \big((1,0)E_{11}\big)_{11} \neq (1,1),$$

and by Proposition 2.1, $\theta$ is not in the form given in Theorem 3.9. Consequently, all conditions in Theorem 3.9 are fulfilled except the condition $(*)$. This is because, if condition $(*)$ holds, then $\theta$ must be in the form $\varphi_D \circ \Lambda$, as desired in Theorem 3.9.

## 4. Skew polynomial modules

In this section, motivated the previous definitions and results, we introduce the notion of skew polynomial module as a generalization of skew polynomial ring and present some results and examples concerning this definition.

**Definition 4.1.** Let $M$ be an $(R,S)$-bimodule, $\alpha : R \to R$, $\beta : S \to S$ be ring endomorphisms and $\gamma : M \to M$ be a bimodule homomorphism relative to $(\alpha, \beta)$. Suppose that $\delta_R : R \to R$ is an $\alpha$-derivation, $\delta_S : S \to S$ is a $\beta$-derivation and $\tau : M \to M$ is a generalized $\gamma$-derivation relative to $(\delta_R, \delta_S)$. We shall call an $(R[x;\alpha,\delta_R], S[y;\beta,\delta_S])$-bimodule $M'$ a *skew polynomial module* $M[y;\gamma,\tau]$ *over* $M$ if

(i) $M'$ contains $M$ as an $(R,S)$-subbimodule,
(ii) for each $m \in M$, we have $xm = \gamma(m)y + \tau(m)$,
(iii) each element $p \in M[y;\gamma,\tau]$ is uniquely written as $p = m_0 + m_1 y + \cdots + m_k y^k$, with $m_i \in M$ and $y^i \in S[y;\beta,\delta_S]$ $(1 \leqslant i \leqslant k)$.

**Remark 4.2.** Let $M$ be an $(R,S)$-bimodule and assume that the conditions of above definition hold. Then $\theta(r,m,s) = (\alpha(r), \gamma(m), \beta(s))$ is a ring endomorphism of $T = \mathcal{T}(R,M,S)$ and $d(r,m,s) = (\delta_R(r), \tau(m), \delta_S(s))$ is a $\theta$-derivation of $T$. Since the skew polynomial ring $T[z;\theta,d]$ exists [11, Proposition 2.3], so by Proposition 3.6, the skew polynomial module $M[y;\gamma,\tau]$ always exists.

Next, we show that the module $M[y;\gamma,\tau]$ is unique up to isomorphism.

**Theorem 4.3.** *Let $M$ be an $(R,S)$-bimodule, $\gamma : M \to M$ be a bimodule homomorphism relative to $(\alpha,\beta)$, $\tau : M \to M$ be a generalized $\gamma$-derivation relative to $(\delta_R,\delta_S)$, and $M[y;\gamma,\tau]$ be a skew polynomial module over $M$. Suppose that we have a $(R[x;\alpha,\delta_R], S[y;\beta,\delta_S])$-bimodule $N$, and an $(R,S)$-bimodule homomorphism $\phi : M \to N$ such that $x\phi(m) = \phi(\gamma(m))y + \phi(\tau(m))$ for all $m \in M$. Then there exists a unique $(R[x;\alpha,\delta_R], S[y;\beta,\delta_S])$-bimodule homomorphism $\Phi : M[y;\gamma,\tau] \to N$ such that $\Phi|_M = \phi$.*

**Proof.** Define $\bar{\phi} : \mathcal{T}(R,M,S) \to \mathcal{T}(R[x;\alpha,\delta_R], N, S[y;\beta,\delta_S])$, by

$$\bar{\phi}(r,m,s) = \big(r, \phi(m), s\big).$$

If $I_R : R \to R[x;\alpha,\delta_R]$ and $I_S : S \to S[y;\beta,\delta_S]$ are given by $I_R(r) = r$ and $I_S(s) = s$, then $\phi$ is a bimodule homomorphism relative to $(I_R, I_S)$. So, by Corollary 2.2, $\bar{\phi}$ is a ring homomorphism. Also, $\theta(r,m,s) = (\alpha(r), \gamma(m), \beta(s))$ is a ring endomorphism of $T = \mathcal{T}(R,M,S)$ and $d(r,m,s) = (\delta_R(r), \tau(m), \delta_S(s))$ is a $\theta$-derivation of $T$. Let $(x,0,y) \in \mathcal{T}(R[x;\alpha,\delta_R], N, S[y;\beta,\delta_S])$. We have

$$\begin{aligned}
(x,0,y)\bar{\phi}(r,m,s) &= \big(xr, x\phi(m), ys\big) \\
&= \big(\alpha(r)x + \delta_R(r), \phi\big(\gamma(m)\big)y + \phi\big(\tau(m)\big), \beta(s)y + \delta_S(s)\big) \\
&= \big(\alpha(r), \phi\big(\gamma(m)\big), \beta(s)\big)(x,0,y) + \big(\delta_R(r), \phi\big(\tau(m)\big), \delta_S(s)\big).
\end{aligned}$$

Thus

$$(x,0,y)\bar{\phi}(r,m,s)=\bar{\phi}\big(\theta(r,m,s)\big)(x,0,y)+\bar{\phi}\big(d(r,m,s)\big).$$

Now, by Lemma 3.1 and the ring isomorphism in Proposition 3.6, we have the following unique ring homomorphism:

$$\psi:\mathcal{T}\big(R[x;\alpha,\delta_R],M[y;\gamma,\tau],S[y;\beta,\delta_S]\big)\to\mathcal{T}\big(R[x;\alpha,\delta_R],N,S[y;\beta,\delta_S]\big),$$

where $\psi|_T=\bar{\phi}$ and $\psi(x,0,y)=(x,0,y)$. Therefore we have $\psi(r,m,s)=(r,\phi(m),s)$ and hence $\psi(E_{11})=E_{11}$ and $\psi(E_{22})=E_{22}$. So, by Corollary 2.2, $\psi$ is of the from

$$\psi(q_1,p,q_2)=\big(\bar{\phi}_1(q_1),\Phi(p),\bar{\phi}_2(q_2)\big),$$

where $\bar{\phi}_1:R[x;\alpha,\delta_R]\to R[x;\alpha,\delta_R]$ and $\bar{\phi}_2:S[y;\beta,\delta_S]\to S[y;\beta,\delta_S]$ are ring endomorphisms and $\Phi:M[y;\gamma,\tau]\to N$ is a bimodule homomorphism relative to $(\bar{\phi}_1,\bar{\phi}_2)$. Also from above relations we have $\bar{\phi}_1(r)=r$ and $\bar{\phi}_1(x)=x$. So by Lemma 3.1, $\bar{\phi}_1:R[x;\alpha,\delta_R]\to R[x;\alpha,\delta_R]$ must be the identity map, and by a similar argument, $\bar{\phi}_2:S[y;\beta,\delta_S]\to S[y;\beta,\delta_S]$ is also the identity map. Hence we have

$$\Phi(q_1p)=\bar{\phi}_1(q_1)\Phi(p)=q_1\Phi(p)\quad\text{and}\quad\Phi(pq_2)=\Phi(p)\bar{\phi}_2(q_2)=\Phi(p)q_2,$$

for each $q_1\in R[x;\alpha,\delta_R]$, $q_2\in S[y;\beta,\delta_S]$ and $p\in M[y;\gamma,\tau]$.

So $\Phi$ is an $(R[x;\alpha,\delta_R],S[y;\beta,\delta_S])$-bimodule homomorphism and we have

$$\Phi(m)E_{12}=\psi(mE_{12})=\phi(m)E_{12},$$

for each $m\in M$. Therefore, $\Phi(m)=\phi(m)$, for each $m\in M$; i.e., $\Phi|_M=\phi$. Now, if $\Phi':M[y;\gamma,\tau]\to N$ is an $(R[x;\alpha,\delta_R],S[y;\beta,\delta_S])$-bimodule homomorphism such that $\Phi'|_M=\phi$, then the mapping:

$$\psi':\mathcal{T}\big(R[x;\alpha,\delta_R],M[y;\gamma,\tau],S[y;\beta,\delta_S]\big)\to\mathcal{T}\big(R[x;\alpha,\delta_R],N,S[y;\beta,\delta_S]\big),$$

given by $\psi'(q_1,p,q_2)=(q_1,\Phi'(p),q_2)$ is easily seen to be a ring homomorphism such that

$$\psi'(r,m,s)=\big(r,\phi(m),s\big)=\bar{\phi}(r,m,s)\quad\text{and}\quad\psi'(x,0,y)=(x,0,y).$$

So by the uniqueness of $\psi$, we must have $\psi'=\psi$ and hence $\Phi'=\Phi$. Therefore, $\Phi$ is unique. □

**Corollary 4.4.** *Let $M$ be an $(R,S)$-bimodule, $\gamma:M\to M$ be a bimodule homomorphism relative to $(\alpha,\beta)$, and $\tau:M\to M$ be a generalized $\gamma$-derivation relative to $(\delta_R,\delta_S)$. If $M[y;\gamma,\tau]$ and $M[y;\gamma,\tau]'$ are skew polynomial modules over $M$, there exists a unique $(R[x;\alpha,\delta_R],S[y;\beta,\delta_S])$-bimodule isomorphism $\Lambda:M[y;\gamma,\tau]\to M[y;\gamma,\tau]'$ such that $\Lambda|_M=id_M$.*

**Proof.** Define $\phi:M\to M[y;\gamma,\tau]'$ by $\phi(m)=m$. Then $\phi$ is an $(R,S)$-bimodule homomorphism such that $x\phi(m)=xm=\phi(\gamma(m))y+\phi(\tau(m))$. So $\phi$ satisfies the conditions of Theorem 4.3, hence there exists a unique $(R[x;\alpha,\delta_R],S[y;\beta,\delta_S])$-bimodule homomorphism $\Lambda:M[y;\gamma,\tau]\to M[y;\gamma,\tau]'$ such that $\Lambda(m)=m$ for each $m\in M$. Similarly, by Theorem 4.3 there exists a unique $(R[x;\alpha,\delta_R],S[y;\beta,\delta_S])$-bimodule homomorphism $\Lambda':M[y;\gamma,\tau]'\to M[y;\gamma,\tau]$ such that $\Lambda'(m)=m$ for each $m\in M$. For any $p\in M[y;\gamma,\tau]'$ we have

$$\begin{aligned}\Lambda \circ \Lambda'(p) &= \Lambda \circ \Lambda'\left(\sum_i m_i y^i\right) = \Lambda\left(\sum_i \Lambda'(m_i) y^i\right)\\ &= \Lambda\left(\sum_i m_i y^i\right) = \sum_i m_i y^i\\ &= p.\end{aligned}$$

So $\Lambda \circ \Lambda' = id_{M[y;\gamma,\tau]'}$ and similarly $\Lambda' \circ \Lambda = id_{M[y;\gamma,\tau]}$. Therefore $\Lambda$ is an $(R[x;\alpha,\delta_R], S[y;\beta,\delta_S])$-bimodule isomorphism such that $\Lambda(m) = m$ for each $m \in M$. The uniqueness of $\Lambda$ follows from Theorem 4.3. □

Observe that in Theorem 4.3 the bimodule homomorphism $\Phi$ is defined by $\Phi(\sum_i m_i y^i) = \sum_i \phi(m_i) y^i$ and hence in Corollary 4.4 the bimodule isomorphism is given by $\Lambda(\sum_i m_i y^i) = \sum_i m_i y^i$.

We also observe that by Corollary 4.4 the skew polynomial module $M[y;\gamma,\tau]$ is unique up to isomorphism.

By considering the appropriate generalized $\gamma$-derivations on a bimodule $M$, several types of skew polynomial modules over $M$ can be obtained. In the sequel we provide several examples of skew polynomial modules.

**Example 4.5.** Let $R$ be a ring, $\alpha$ be a ring endomorphism of $R$, and $\delta$ be an $\alpha$-derivation of $R$. Considering $R$ as an $(R,R)$-bimodule, $\alpha$ is a bimodule homomorphism relative to $(\alpha,\alpha)$ and $\delta$ is a generalized $\alpha$-derivation relative to $(\delta,\delta)$. So we have the skew polynomial module $N := R[x;\alpha,\delta]$. On the other hand, $R[x;\alpha,\delta]$, as an $(R[x;\alpha,\delta], R[x;\alpha,\delta])$-bimodule, satisfies the conditions in Definition 4.1 via $\alpha$ and $\delta$. Therefore, $R[x;\alpha,\delta]$ is a skew polynomial module over $R$ and hence $N$ is isomorphic to $R[x;\alpha,\delta]$, as $(R[x;\alpha,\delta], R[x;\alpha,\delta])$-bimodule, by Corollary 4.4.

From this example we see that the notion of skew polynomial modules generalize the concept of skew polynomial rings.

**Example 4.6.** Let $M$ be a $(R,S)$-bimodule , $\alpha = id_R$, $\beta = id_S$ and $\gamma = id_M$. Suppose that $\delta_R : R \to R$ and $\delta_S : S \to S$ are derivations and $\tau : M \to M$ is a generalized derivation relative to $(\delta_R, \delta_S)$. Then the skew polynomial module $M[y;\gamma,\tau]$ is the differential polynomial module $M[y;\tau]$ which was introduced in [9]. Note that $M[y;\tau]$ is an $(R[x;\delta_R], S[y;\delta_S])$-bimodule.

**Example 4.7.** Let $M$ be an $(R,S)$-bimodule and $\gamma : M \to M$ be a bimodule homomorphism relative to $(\alpha,\beta)$. Suppose that $\delta_R : R \to R$, $\delta_S : S \to S$ and $\tau : M \to M$ are zero mappings. Then the skew polynomial module $M[y;\gamma,\tau]$ is an $(R[x;\alpha], S[y;\beta])$-bimodule which is denoted by $M[y;\gamma]$ and in this case, for each $m \in M$, we have $xm = \gamma(m)y$. Moreover, if $\alpha = id_R$, $\beta = id_S$ and $\gamma = id_M$, then the skew polynomial module $M[y;\gamma]$ is denoted by $M[y]$ which is an $(R[x], S[y])$-bimodule and $xm = my$, for each $m \in M$.

Recently, several types of polynomial modules over right modules have been introduced (for instance, see [1,3,15] and [16]). As will be shown in the following example, the skew polynomial modules cover these polynomial modules.

**Example 4.8.** Let $M$ be a right $S$-module. Denote the ring of all $S$-module endomorphisms $End_S(M)$ of $M$ by $R$. Then $M$ is an $(R,S)$-bimodule equipped with $\varphi \cdot m := \varphi(m)$ ($m \in M$, $\varphi \in R$), because $\varphi \cdot (ms) = \varphi(ms) = \varphi(m)s = (\varphi \cdot m)s$, for all $s \in S$, $m \in M$ and $\varphi \in R$.

(i) If we consider $M[y]$ as in Example 4.7, then $M[y]$ is a skew polynomial module which is an $(R[x], S[y])$-bimodule and each element $p \in M[y]$ is uniquely written as $p = m_0 + m_1 y + \cdots + m_k y^k$, with $m_i \in M$ and $y^i \in S[y]$ ($0 \leqslant i \leqslant k$). We see that the right $S[y]$-module $M[y]$ covers the notion of right $S[y]$-module $M[y]$, as introduced in [3].

(ii) Suppose $\beta : S \to S$ is a ring endomorphism and $\alpha = id_R$. Then the zero map $\gamma : M \to M$ is a bimodule homomorphism relative to $(\alpha, \beta)$. If $\delta_R : R \to R$, $\delta_S : S \to S$ and $\tau : M \to M$ are zero mappings, then $\tau$ is a generalized $\gamma$-derivation relative to $(\delta_R, \delta_S)$ and $M[y; \gamma, \tau]$ is an $(R[x], S[y; \beta])$-bimodule. In view of the definition of skew polynomial modules, the right $S[y; \beta]$-module $M[y; \gamma, \tau]$ covers the notion of right $S[y; \beta]$-module $M[y]$, as introduced in [16] and [1].

(iii) Given a derivation $\delta_S : S \to S$, assume that $\alpha = id_R$, $\beta = id_S$, $\gamma = id_M$, $\delta_R = 0$ and $\tau = 0$. So $\gamma$ is a bimodule homomorphism relative to $(\alpha, \beta)$ and $\tau$ is a generalized $\gamma$-derivation relative to $(\delta_R, \delta_S)$. Therefore, $M[y; \gamma, \tau]$ is an $(R[x], S[y; \delta])$-bimodule which, as a right $S[y; \delta]$-module, covers the notion of right $S[y; \delta]$-module $M[y]$, for as introduced in [15].

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## References

[1] N. Agatev, S. Halicioğlu, A. Harmanci, On reduced modules, Commun. Fac. Sci. Univ. Ank. Ser. A1 58 (2009) 9–16.
[2] M. Auslander, M.I. Platzeck, I. Reiten, Coxeter functors without diagrams, Trans. Amer. Math. Soc. 250 (1979) 1–46.
[3] M. Başer, M.T. Koşan, On Armendariz and quasi-Armendariz modules, Taiwanese J. Math. 12 (3) (2008) 573–582.
[4] G.F. Birkenmeier, H.E. Heatherly, J.Y. Kim, J.K. Park, Triangular matrix representations, J. Algebra 230 (2000) 558–595.
[5] G.F. Birkenmeier, J.K. Park, Triangular matrix representations of ring extensions, J. Algebra 265 (2003) 457–477.
[6] S.U. Chase, A generalization of the ring of triangular matrices, Nagoya Math. J. 18 (1961) 13–25.
[7] V. Dlab, C.M. Ringel, Representations of graphs and algebras, Mem. Amer. Math. Soc. 173 (1976).
[8] R.M. Fossum, P.A. Griffith, I. Reiten, Trivial Extensions of Abelian Categories, Lecture Notes in Math., vol. 456, Springer-Verlag, Berlin, Heidelberg, New York, 1975.
[9] H. Ghahramani, A. Moussavi, Differential polynomial rings of triangular matrix rings, Bull. Iranian Math. Soc. 34 (2) (2008) 71–96.
[10] M.N. Ghosseiri, The structure of $(\alpha, \beta)$-derivation of triangular rings, Iran. J. Sci. Technol. Trans. A 29 (A3) (2005) 507–514.
[11] K.R. Goodearl, R.B. Warfield, An Introduction to Noncommutative Noetherian Rings, second edition, Cambridge University Press, Cambridge, 2004.
[12] E.L. Green, The representation theory of tensor algebras, J. Algebra 34 (1975) 136–171.
[13] E.L. Green, I. Reiner, Integral representations and diagrams, Michigan Math. J. 25 (1978) 53–84.
[14] Dong Han, Feng Wei, Jordan $(\alpha, \beta)$-derivations on triangular algebras and related mappings, Linear Algebra Appl. 434 (2011) 259–284.
[15] E. Hashemi, On $\delta$-quasi Armendariz modules, Bull. Iranian Math. Soc. 33 (2) (2007) 15–26.
[16] T.K. Lee, Y. Zhou, Reduced modules, in: Rings, Modules, Algebras, and Abelian Groups, in: Lect. Notes Pure Appl. Math., vol. 236, Marcel Dekker, New York, 2004, pp. 365–377.
[17] I. Reiten, Stable equivalence of self-injective algebras, J. Algebra 40 (1976) 63–74.
[18] C.M. Ringel, Representations of k-spaces and bimodules, J. Algebra 41 (1976) 269–302.
[19] Zhongkui Liu, Xiaoyan Yang, Triangular matrix representations of skew monoid rings, Math. J. Okayama Univ. 52 (2010) 97–109.